Back extrusion of Vočadlo (Robertson-Stiff) fluids - semi-analytical solution

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ABSTRACT

Back extrusion method determining rheological characteristics of fluids is based on plunging of a circular rod into an axisymmetrically located circular cup containing the experimental sample. The aim is to present a procedure calculating the individual rheological parameters appearing in the Vočadlo model - yield stress, consistency parameter and flow behaviour index.

INTRODUCTION

At present standard rheometers provide sufficiently precise measurements characterising behaviour of non-Newtonian materials. In practice, this accuracy is not always necessary, and the methods providing relatively cheap, fast and sufficient measurements of the rheological characteristics are fully acceptable.

Back extrusion problem (Steffe and Osorio[1]) - when sample compression causes material to flow through the annulus formed between the plunger (cylindrical rod) and the cylinder container (see Fig.1) - represents one of these methods and appears in various industrial branches, e.g. metal processing, petroleum industry, food processing.

Determination of the model parameters using a back extrusion technique has been hitherto derived for two models - two-parameter (2P) power-law one with a consistency index K and a flow behaviour index n

$$\tau = K\left|\dot{\gamma}\right|^{n-1} \cdot \dot{\gamma} \qquad (1)$$

and three-parameter (3P) Herschel-Bulkley one

$$\tau = \left(K\left|\dot{\gamma}\right|^{n-1} + \frac{\tau_0}{\left|\dot{\gamma}\right|} \right)\dot{\gamma} \qquad \text{for} \quad \left|\tau\right| \geq \tau_0 \qquad (2)$$

$$\dot{\gamma} = 0 \qquad \text{for} \quad \left|\tau\right| \leq \tau_0 \qquad (3)$$

taking into account viscoplastic behaviour of the materials tested through a yield stress τ_0.

Osorio and Steffe[2] derived an analytical expression for a determination of flow behaviour index n in the power-law model (1). This expression is based on knowledge of a force corrected for buoyancy (provided by a compression testing machine such as those manufactured by Instron Corp.), length of an immersed plunger and its velocity for two successive runs with different plunger velocities. Prior to a determination of consistency parameter K it is necessary to calculate a location λ of zero shear stress in an annulus. This is the only numerical step in the whole procedure that is possible to bypass using the tabulated values of λ for individual combinations of flow behaviour index n and annular aspect ratio κ, or directly compute a location λ solving a simple integral equation (see Osorio and Steffe[2]) analogous to that presented in Hanks and Larsen[3] for the case of a stationary inner cylinder and pressure gradient exerted in the axial direction.

Determination of the model parameters for fluids obeying the Herschel-Bulkley model is not so straightforward as in the preceding case. Osorio and Steffe[4] derived a procedure how to determine all three parameters, they provide diagrams enabling approximation of the concrete values. At present, with the development of common computational possibilities, it is more advantageous to use the equations they derived and compute the values of the individual parameters numerically.

Barnes and Walters[5] launched an ample discussion concerning possible interpretation of the meaning (or existence) of the notion 'yield stress'. This discussion was summarised in the paper by Barnes[6]. Reflecting this discussion and also viewpoints presented by Corradini and Peleg[7] it seems that the 3P Vočadlo model[8] (sometimes called Robertson-Stiff one[9])

$$\tau = \left[K\left|\dot{\gamma}\right|^{\frac{n-1}{n}} + \left(\frac{\tau_0}{\left|\dot{\gamma}\right|}\right)^{\frac{1}{n}} \right]^n \dot{\gamma} \quad \text{for} \quad \left|\tau\right| \geq \tau_0 \;, \quad (4)$$

$$\dot{\gamma} = 0 \qquad \text{for} \quad \left|\tau\right| \leq \tau_0 \quad (5)$$

better corresponds to reality and applicability than the 3P Herschel-Bulkley model. The reasons are as follows:

- functional arrangement gives better chance to derive analytical solution for a given problem;
- position of a yield stress τ_0 as a member in rel.4 does not represent so strict singularity as an additive member τ_0 in rel.2;
- flow curve shear stress τ vs. shear rate $\dot{\gamma}$ does not exhibit an infinite slope at $\dot{\gamma} = 0$ as in the case of the Herschel-Bulkley model but attains a finite value, see Fig.2.

The aim of this contribution is to present a procedure how to determine - for materials obeying the Vočadlo model - three corresponding empirical parameters using a back extrusion technique.

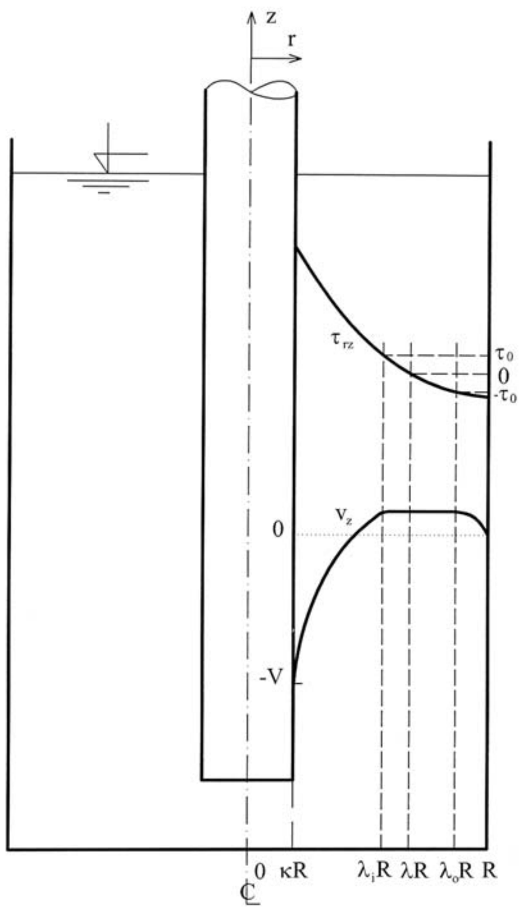

Figure 1. Definition sketch of a back extrusion.

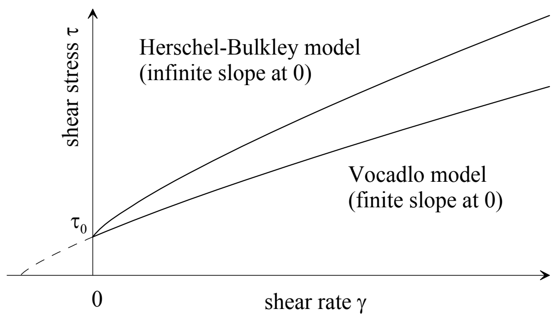

Figure 2. Vočadlo and Herschel-Bulkley models.

PROBLEM FORMULATION

Filip and David[10] presented analysis of axial flow of non-Newtonian fluids obeying the Vočadlo model in concentric annuli when flow is caused simultaneously by the inner cylinder moving along its axis and by the pressure gradient imposed in the axial direction. Both cases - either pressure gradient assists to the moving cylinder or opposes - were considered. All possible cases (six) with respect to the possible positions of the plug flow regions were uniquely diversified through the derived semi-analytical criteria using the entry (geometrical, kinematical and rheological) parameters. For each possible case there was derived the explicit semi-analytical expression for the volumetric flow rate.

Out of these six cases the only one takes place in the description of a back extrusion problem. In the following there is supposed that the flow is steady, laminar, incompressible, isothermal and axial with negligible end effects of the cylinders. The last assumption was studied and justified in Osorio et al.[11].

The Vočadlo model rewritten in the form corresponding to the flow situation in a back extrusion (see Fig.1) is of the form

$$\tau_{rz} = \left[K^{\frac{1}{n}} \left| \frac{dv_z}{dr} \right|^{\frac{n-1}{n}} + \tau_0^{\frac{1}{n}} \left| \frac{dv_z}{dr} \right|^{-\frac{1}{n}} \right]^n \frac{dv_z}{dr} \qquad \text{for } |\tau_{rz}| \geq \tau_0 \quad , \tag{6}$$

$$\frac{dv_z}{dr} = 0 \qquad \text{for } |\tau_{rz}| \leq \tau_0 \quad . \tag{7}$$

Introducing the following dimensionless transformations (for notation see Figs.1,3, rels.6,7, q denotes volumetric flow rate, V represents velocity of a plunger)

$$\xi = \frac{r}{R}, \quad \varphi = \frac{v_z}{V}, \quad T = \frac{2\tau_{rz}}{|P|R}, \quad T_0 = \frac{2\tau_0}{|P|R}, \quad \Lambda = \frac{|P|R}{2K}\left(\frac{R}{V}\right)^n, \quad Q = \frac{q}{2\pi R^2 V} \tag{8}$$

the problem of flow within an annulus can be reformulated in the form

$$T = \frac{\lambda^2}{\xi} - \xi \quad , \tag{9}$$

$$\varphi(\kappa) = -1 \quad , \quad \varphi(1) = 0 \quad , \tag{10}$$

$$T = \left[\Lambda^{-s} \left| \frac{d\varphi}{d\xi} \right|^{1-s} + T_0^{-s} \left| \frac{d\varphi}{d\xi} \right|^{-s} \right]^n \frac{d\varphi}{d\xi} \qquad \text{for } |T| \geq T_0 , \tag{11}$$

$$\frac{d\varphi}{d\xi} = 0 \qquad \text{for } |T| \leq T_0 \tag{12}$$

where λ^2 is a dimensionless constant of integration, $s=1/n$.

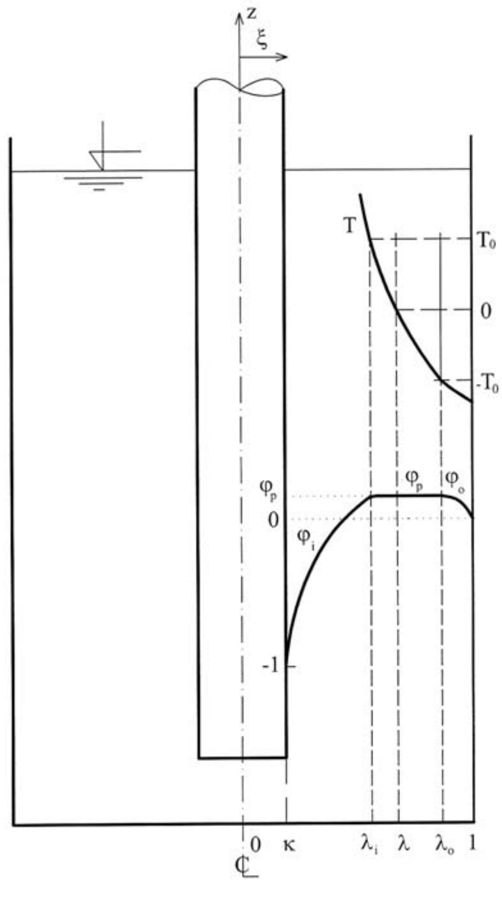

Figure 3. Definition sketch of a back extrusion after dimensionless transformations.

If λ_i, λ_o denote the dimensionless boundary values of the plug flow region (see Fig.3), then from Eq.9 it follows that

$$\lambda^2 = \lambda_i \lambda_o \quad , \tag{13}$$

$$\lambda_i = \lambda_o - T_0 \quad . \tag{14}$$

For simplification the following notation will be used in the further analysis

$$H(\xi) = \left| \xi - \frac{\lambda_i(\lambda_i + T_0)}{\xi} \right|^s \quad . \tag{15}$$

The solution of the above stated problem provides the following expressions for the inner, plug-flow region and outer velocity profiles

$$\frac{d\varphi_i}{d\xi} = \Lambda^s \left[\left(\frac{\lambda^2}{\xi} - \xi \right)^s - T_0^s \right]$$

$$\text{for } \kappa \le \xi < \lambda_i \text{ (where } \frac{d\varphi}{d\xi} > 0 \text{)} \ , \tag{16}$$

$$\frac{d\varphi_p}{d\xi} = 0 \quad \text{for} \quad \lambda_i \le \xi \le \lambda_o \quad , \tag{17}$$

$$\frac{d\varphi_o}{d\xi} = -\Lambda^s \left[\left(\xi - \frac{\lambda^2}{\xi} \right)^s - T_0^s \right]$$

$$\text{for} \quad \lambda_o < \xi \le 1 \text{ (where } \frac{d\varphi}{d\xi} < 0 \text{)} \ . \tag{18}$$

From the condition of continuity of the velocity profile

$$\varphi_i(\lambda_i) = \varphi_o(\lambda_o) \tag{19}$$

it follows that λ_i is a solution of the equation

$$\begin{aligned} &\int_\kappa^{\lambda_i} \Lambda^s H(\xi) d\xi + \int_{\lambda_i + T_0}^{1} \Lambda^s H(\xi) d\xi - \\ &-(2\lambda_i + T_0 - \kappa - 1)\Lambda^s T_0^s - 1 = 0 \end{aligned} \tag{20}$$

If we compare a volumetric flow rate q through an annulus as given by rel.8 and visually in Fig.2, we get

$$2\pi R^2 V Q = \pi (\kappa R)^2 V \quad . \tag{21}$$

From here it follows that

$$Q = \kappa^2 / 2 \quad . \tag{22}$$

As the determination of dimensionless flow rate Q is basically similar to that derived in Malik and Shenoy[12] for power-law fluids, in the following we only introduce the final result

$$\begin{aligned} Q = &-\frac{1}{2}\left(\frac{1-s}{3+s}\lambda^2 - \kappa^2 \right) - \\ &-\left[\begin{matrix} \dfrac{1+\kappa^3 - \lambda_i^3 - \lambda_o^3}{6} - \\ -\dfrac{1-s}{2(3+s)}\lambda^2 (1 + \kappa - \lambda_i - \lambda_o) \end{matrix} \right] \Lambda^s T_0^s + \\ &+ \frac{\Lambda^s}{2(3+s)} \left[\begin{matrix} (1-\lambda^2)^{1+s} - \lambda_o^{1-s}(\lambda_o^2 - \lambda^2)^{1+s} + \\ +\lambda_i^{1-s}(\lambda^2 - \lambda_i^2)^{1+s} - \\ -\kappa^{1-s}(\lambda^2 - \kappa^2)^{1+s} \end{matrix} \right] \end{aligned} \tag{23}$$

Comparing rels.22,23 we obtain

$$\begin{aligned} &-\frac{1-s}{2(3+s)}\lambda^2 - \\ &-\left[\begin{matrix} \dfrac{1+\kappa^3 - \lambda_i^3 - \lambda_o^3}{6} - \dfrac{1-s}{2(3+s)}\lambda^2 \times \\ \times (1 + \kappa - \lambda_i - \lambda_o) \end{matrix} \right] \Lambda^s T_0^s + \\ &+ \frac{\Lambda^s}{2(3+s)} \times \\ &\times \left[\begin{matrix} (1-\lambda^2)^{1+s} - \lambda_o^{1-s}(\lambda_o^2 - \lambda^2)^{1+s} + \\ +\lambda_i^{1-s}(\lambda^2 - \lambda_i^2)^{1+s} - \kappa^{1-s}(\lambda^2 - \kappa^2)^{1+s} \end{matrix} \right] = 0 \end{aligned} \tag{24}$$

PROBLEM SOLUTION

First, out of three empirical parameters appearing in the Vočadlo model, a yield stress τ_0 will be determined. As this step is the same as that introduced by Osorio and Steffe[4] for the determination of yield stress in the Herschel-Bulkley model, in the following we use a notation used in that paper.

Let us denote F_T a static force at the base of the plunger formed successively by a friction force along the plunger F_f, force responsible for fluid flow in the upward direction F_u, and buoyancy force F_b

$$F_T = F_f + F_u + F_b \tag{25}$$

$$F_T = 2\pi\kappa R L \tau_w + \pi\left(\kappa R\right)^2 \Delta P + \rho g L \pi \left(\kappa R\right)^2 \tag{26}$$

where L represents the length of a plunger penetrated into liquid; ΔP is a difference between pressures at the entrance to annulus and at the plunger base; ρ stands for fluid density; g is a gravity acceleration.

When the plunger is stopped (i.e. $\varphi \equiv 0$) a static force F_T attain a value F_{T_e}

$$F_{T_e} = 2\pi\kappa R L \tau_0 + \rho g L \pi \left(\kappa R\right)^2 \quad . \tag{27}$$

From here it follows that

$$\tau_0 = \frac{F_{T_e} - \rho g L \pi \left(\kappa R\right)^2}{2\pi\kappa R L} \quad , \tag{28}$$

force F_{T_e} is experimentally recorded after the plunger is stopped.

From rels. 25,26 we obtain

$$\frac{F_T - \rho g L \pi \left(\kappa R\right)^2}{\pi L \kappa R^2 \left|P\right|} = T_w + \kappa \tag{29}$$

From the experimental data we know a value for F_T (force recorded just before the plunger is stopped) and hence rel.29 provides a value for T_w. Consequently we determine λ^2 from rel.9 written at the point $\xi=\kappa$:

$$\lambda^2 = \kappa\left(1 + T_w\right) \tag{30}$$

Eqs.13,14 provide the values for λ_i, λ_o as T_0 and λ^2 are known.

Finally, the two remaining empirical parameters K, n in the Vočadlo model can be calculated from Eqs. 20 and 24.

DISCUSSION

The application of the whole procedure presented above significantly subjects to the assumption of an axisymmetrical position of a plunger with respect to a cylinder container. Deviation from this assumption can cause non-negligible errors in the prediction of the parameters τ_0, K, n.

As the power-law model and the Bingham model are the sub-cases of the Vočadlo model for τ_0=0 and n=1, respectively, it is also possible to apply the procedure presented above to these two models with the corresponding presetting of the individual parameters.

CONCLUSION

Determination of all three parameters in the Vočadlo model with use of a back extrusion technique represents a cheap and time-saving experimental method only requiring a compression testing machine and a common commercial software enabling the calculation of these parameters. The accuracy of these parameters does not attain the one when the sophisticated rheometers are used, nevertheless from the practical point of view is - in many applications - fully satisfactory.

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