Taylor-Couette Instability of Giesekus Fluids

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ABSTRACT

Taylor-Couette instability of Giesekus fluids is investigated at large gaps using a temporal, linear instability analysis assuming that the perturbations are axisymmetric. Using normal-mode concept, an eigenvalue problem is obtained which is solved numerically using pseudo-spectral collocation method. A fluid's elasticity is predicted to have a stabilizing or destabilizing effect on the Couette flow depending on the Weissenberg number being smaller or larger than a critical value.

INTRODUCTION

Tangential flow between two infinitely-long concentric cylinders rotating at different angular velocities (the circular Couette flow) loses its stability and switches to more complicated laminar flow patterns depending on the Reynolds number[1]. The first mode of instability exhibits itself as a secondary flow comprising a pair of stationary-mode, axisymmetric, counter-rotating, toroidal roll cells stacked in the gap. Taylor[2] relied on a normal-mode, linear instability analysis to show that for Newtonian fluids, when the gap spacing is narrow, the secondary flow should emerge whenever the parameter $\mathrm{Re}\sqrt{d/R_1}$ (the so-called Taylor number) exceeds 41, where d is the gap spacing, R_1 is the radius of the inner cylinder, and Re is the Reynolds number (based on the angular velocity of the inner cylinder). The fact that the critical Taylor number obtained by Taylor[2] closely matched the experimental value was regarded by many as one of the greatest achievements of the linear instability analysis. In ensuing works, other aspects of the flow (e.g., the effect of the gap size, and the effect of the cylinders eccentricity) has been investigated[3]. There are also many works addressing the effects of a fluid's non-Newtonian behavior on the instability picture of the circular Couette flow[4-9].

The interest in studying Taylor-Couette instability for non-Newtonian fluids stems mainly from the fact that Couette rheometers are widely used for characterizing non-Newtonian fluids. But, the rheological data obtained in Couette rheometers are useful only if the gap is free from secondary flow. Therefore, determining conditions under which secondary flow occurs in the gap is very important in shear rheometry. This is perhaps why the literature is so rich when it comes to the Taylor-Couette instability of viscoelastic fluids[5-9]. One can notably mention the numerical work carried out by Beris[10] in which the critical Taylor number has been calculated for the Giesekus fluid under narrow-gap restriction.

In the present work, we are going to extend Beris' work[10] to the large-gap situation, to the best of our knowledge, for the first time. The case of large gap is more common in shear rheometry, particularly when dealing with fluids containing solid particles, or for highly-viscous fluids. To that end, we superimpose an infinitesimal perturbation, represented by normal modes, to the base flow and see what happens to its amplitude in the course of time. An eigenvalue problem will be obtained this way in which terms nonlinear in the perturbation quantities will be dropped. To ensure high degree of accuracy, spectral method will be used to find the critical conditions for the onset of instability.

MATHEMATICAL FORMULATIONS

Taylor-Couette geometry basically consists of two infinitely long, concentric cylinders of radii R_1 and R_2 with the fluid confined in the annulus between them. In general, the cylinders are rotating independently with an angular velocity Ω_1 and Ω_2 either in the same direction or in opposite direction. For non-Newtonian fluids, the equations to start with are the Cauchy equations of motion together with the continuity equation. Assuming that the fluid is incompressible, in polar coordinate system, we have,

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right] - \frac{\partial p}{\partial r} \tag{1a}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = \left[\frac{1}{r^2}\frac{\partial (r^2\tau_{r\theta})}{\partial r} + \frac{\partial (\tau_{z\theta})}{\partial z}\right] \tag{1b}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right) = \left[\frac{1}{r}\frac{\partial (r\tau_{rz})}{\partial r} + \frac{\partial (\tau_{zz})}{\partial z}\right] - \frac{\partial p}{\partial z} \tag{1c}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0 \tag{1d}$$

where it has tacitly been assumed that the flow is axisymmetric (i.e., $\partial/\partial\theta = 0$). In these equations, ρ is the density, v_i represents the velocity components, and τ_{ij} are the components of the stress tensor. The stress components appearing in these equations need a constitutive equation to be related to the velocity field. In the present work, we assume that the fluid of interest obeys the Giesekus model as its constitutive equation. In this viscoelastic fluid model, the extra stress tensor is given by[11],

$$\underset{\approx}{\tau} + \frac{\alpha\lambda}{\eta}(\underset{\approx}{\tau}.\underset{\approx}{\tau}) + \lambda\left[\frac{\partial \underset{\approx}{\tau}}{\partial t} + \left(\vec{V}.\nabla\right)\underset{\approx}{\tau}\right] - \lambda\left(\underset{\approx}{\tau}.\nabla\vec{V} + \nabla\vec{V}^{T}.\underset{\approx}{\tau}\right) = 2\eta\underset{\approx}{D} \tag{2}$$

where 2D is the deformation-rate tensor, α is the mobility, λ is the longest relaxation time of the fluid, and η is the fluid's zero-shear viscosity. It needs to be mentioned that, for $\alpha = 0$ the Giesekus model reduces simply to the upper-convected Maxwell (UCM) model. On the other hand for $\lambda = 0 = \alpha$ the Giesekus model reduces to the Newtonian fluid model.

The Giesekus model is widely regarded as one of the best rheological models for representing polymeric liquids (i.e., polymer melts and concentrated polymer solutions). The mobility factor, which accounts for anisotropic relaxation of polymer macromolecules, profoundly affects the shear- and extensional-flow behaviour of the fluid. In fact, by increasing α (starting from zero) shear-thinning starts at lower shear rates, and the asymptotic value of the extensional viscosity is decreased. This fluid model also correctly predicts the first and second normal stress differences for polymeric liquids provided that the mobility factor is selected in the range of $0 < \alpha \le 0.5$.

BASIC FLOW

As the first step in our instability analysis, we need the basic (i.e., steady) flow velocity and stress fields. Porjafar[12] has recently shown that for Giesekus fluids, the governing equations can be simplified to the following ordinary differential equations as far as the basic-flow velocity profiles, v_θ^0, is concerned,

$$r\frac{d}{dr}\left(\frac{v_\theta^0}{r}\right)=\frac{\tau_{r\theta}^0+\alpha We\left(\tau_{rr}^0+\tau_{\theta\theta}^0\right)\tau_{r\theta}^0}{(1-\varphi)(1+We\tau_{rr}^0)} \tag{3}$$

where $We=\lambda R_1\Omega_1/d$ is the Weissenberg number. Porjafar[12] has also shown that in this ODE, the basic-flow stress terms τ_{rr}^0 and $\tau_{\theta\theta}^0$ can be related to $\tau_{r\theta}^0$ by simple algebraic equations. As such, a shooting scheme in which $\tau_{r\theta}^0$ is guessed at the inner wall and the velocity profile so-obtained is checked to see if it satisfies the no-slip condition at the outer wall can easily be used to find v_θ^0. Figure 1 shows typical basic-flow velocity profiles obtained this way at a mobility factor of $\alpha=0.2$ for a large-gap radii ratio of $\varphi=R_1/R_2=0.65$ when the cylinders are rotating in the opposite directions.

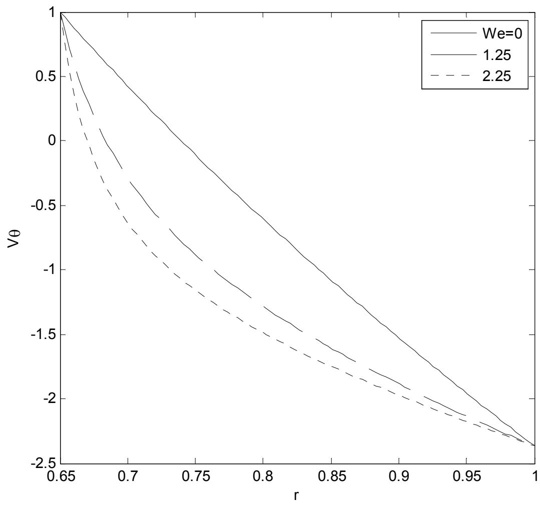

Figure 1. Effect of the Weissenberg number on the basic-flow velocity profiles obtained at $\varphi=0.65,\ \alpha=0.2$ when $\Omega_2/\Omega_1=-1/\varphi$.

Figure 1 shows that the velocity profiles deviate more from a linear one by an increase in the Weissenberg number.

INSTABILITY ANALYSIS

Having calculated the basic-flow velocity and stress fields for the Giesekus fluid, we are now ready to investigate their vulnerability to instability when perturbed slightly. The perturbations are assumed to be axisymmetric and two-dimensional, so that we have.

$$V(r,z,t)=v_\theta^0(r)+\hat{V}(r,z,t), \tag{4}$$

$$\tau(r,z,t)=\tau^0(r)+\hat{\tau}(r,z,t), \tag{5}$$

where $\hat{V}$, and $\hat{\tau}$ represent the perturbation velocity and stress fields. Assuming that the perturbations can be represented by normal modes we have,

$$\hat{V}=\left[A(r)\quad B(r)\quad C(r)\right]e^{ikz+\beta t}, \tag{6}$$

$$\hat{\tau}=\begin{bmatrix} D(r) & E(r) & F(r)\\ E(r) & G(r) & H(r)\\ F(r) & H(r) & Q(r)\end{bmatrix}e^{ikz+\beta t}. \tag{7}$$

In these equations, k is the (dimensionless) real wavenumber, and $\beta=\beta_R+i\beta_I$ is the (dimensionless) complex phase velocity with $\beta_R>0$ signalling the occurrence of instability.

Using continuity equation, C(r) can be related to A(r) by,

$$C(r)=-\frac{1}{ik}\left[\frac{1}{r}\frac{d}{dr}\left(rA(r)\right)\right]. \tag{8a}$$

$$\beta rB(r)=-Re(1-\varphi)A(r)\left[1+rD^*\right]v_\theta^0(r)+ \\ (1-\varphi)\left[2+rD^*\right]E(r)+r(1-\varphi)ikH(r) \tag{8b}$$

$$\beta\left(1-\frac{D^*D^*+\frac{1}{r}D^*-\frac{1}{r^2}}{k^2}\right)A(r)=$$

$$(1-\varphi)\left\{\begin{matrix}2\,\mathrm{Re}\frac{v_\theta^0(r)}{r}B(r)-\left[\frac{G(r)}{r}+D^*Q(r)\right]+\\ \left(\frac{1}{r}+D^*\right)D(r)+\\ ik\left(1+\frac{D^*D^*+\frac{1}{r}D^*-\frac{1}{r^2}}{k^2}\right)F(r)\end{matrix}\right\} \tag{8c}$$

$$\left[1+e\beta+\alpha We\left(\tau_{rr}^0(r)+\tau_{\theta\theta}^0(r)\right)\right]E(r)+$$
$$We(1-\varphi)\left[\left(D^*-\frac{1}{r}\right)\tau_{r\theta}^0(r)-\tau_{r\theta}^0(r)D^*\right]A(r)+$$
$$\alpha We\tau_{r\theta}^0(r)G(r)+We\begin{bmatrix}\alpha\tau_{r\theta}^0(r)-\\(1-\varphi)\left(D^*-\frac{1}{r}\right)v_\theta^0(r)\end{bmatrix}D(r)$$
$$=(1-\varphi)\left[1+We\tau_{rr}^0(r)\right]\left(D^*-\frac{1}{r}\right)B(r) \tag{8d}$$

$$\left[1+e\beta+2\alpha We\tau_{\theta\theta}^0(r)\right]G(r)+$$
$$2We\begin{bmatrix}\alpha\tau_{r\theta}^0(r)-\\(1-\varphi)\left(D^*-\frac{1}{r}\right)v_\theta^0(r)\end{bmatrix}E(r)-$$
$$2We(1-\varphi)\tau_{r\theta}^0(r)\left(D^*-\frac{1}{r}\right)B(r)=$$
$$(1-\varphi)\left[-We\left(D^*-\frac{1}{r}\right)\tau_{\theta\theta}^0(r)+We\frac{\tau_{\theta\theta}^0(r)}{r}+\frac{2}{r}\right]A(r) \tag{8e}$$

$$(1+e\beta)Q(r)=-2(1-\varphi)\left[\frac{1}{r}+D^*\right]A(r) \tag{8f}$$

$$\left[1+e\beta+2\alpha We\tau_{rr}^0(r)\right]D(r)+2\alpha We\tau_{r\theta}^0(r)E(r)$$
$$=(1-\varphi)\left[2\left(1+We\tau_{rr}^0(r)\right)D^*-WeD^*\tau_{rr}^0(r)\right]A(r) \tag{8g}$$

$$\left[1+e\beta+\alpha We\tau_{rr}^0(r)\right]F(r)+\alpha We\tau_{r\theta}^0(r)H(r)=$$
$$(1-\varphi)ik\left\{1+\frac{1+We\tau_{rr}^0(r)}{k^2}\left[D^*D^*+\frac{1}{r}D^*-\frac{1}{r^2}\right]\right\}A(r) \tag{8h}$$

$$\left[1+e\beta+\alpha We\tau_{\theta\theta}^0(r)\right]H(r)+$$
$$We\left[\alpha\tau_{r\theta}^0(r)-(1-\varphi)\left(D^*-\frac{1}{r}\right)\right]F(r)$$
$$=(1-\varphi)ik\left\{B(r)+\frac{We\tau_{r\theta}^0}{k^2}\left[D^*D^*+\frac{1}{r}D^*-\frac{1}{r^2}\right]A(r)\right\} \tag{8i}$$

where $D^*=d/dr$ and $D^*D^*=d^2/dr^2$. From the boundary conditions, one can conclude that: $A=D^*A=0$, and $B=0$.

NUMERICAL METHOD

For solving the instability problem as posed by Eqs. 8a-i, we have decided to rely on the pseudo-spectral, Chebeshev-based, collocation method[13]. The basic idea in all spectral methods is to assume that the unknown function(s) can be approximated by a sum of certain number of base functions; that is,

$$A(y)=\sum_{n=1}^{N-1}a_n\zeta_n(y) \tag{9a}$$

$$B(y)=\sum_{n=1}^{N-1}b_n\xi_n(y) \tag{9b}$$

$$D(y)=\sum_{n=1}^{N-1}d_n\zeta_n(y) \tag{9c}$$

$$E(y)=\sum_{n=1}^{N-1}e_n\xi_n(y) \tag{9d}$$

$$F(y)=\sum_{n=1}^{N-1} f_n \zeta_n(y) \qquad (9e)$$

$$G(y)=\sum_{n=1}^{N-1} g_n \zeta_n(y) \qquad (9f)$$

$$H(y)=\sum_{n=1}^{N-1} h_n \xi_n(y) \qquad (9g)$$

$$Q(y)=\sum_{n=1}^{N-1} q_n \zeta_n(y) \qquad (9h)$$

where $\zeta_n(y)$ and $\xi_n(y)$ are the base functions. In the present work, the base functions will be constructed using Chebyshev polynomials such that they automatically satisfy the boundary conditions. To that end, we write,

$$\zeta_n(y)=T_{n-1}(y)-\frac{2(n+1)}{n+2}T_{n+1}(y)+\frac{n}{n+2}T_{n+3}(y) \qquad (10a)$$

$$\xi_n(y)=-\frac{n-3.15}{2n-5}T_{n-4}(y)-\frac{n-1.85}{2n-5}T_n(y)+T_{n+2}(y) \qquad (10b)$$

where $T_n(y)=\cos\left(n\cos^{-1}(y)\right)$ is the n^{th} Chebyshev polynomials. Since these polynomials are defined in the interval $[-1,+1]$ we have relied on the transformation $y=[2r^*-(1+\varphi)]/(1-\varphi)$ to map the space between the two cylinders to this interval; in this transformation we have: $r^*=\frac{r}{R_2}$. Now, by substituting Eqs. 9a-h into Eqs. 8a-i, an eigenvalue matrix equation is obtained in the form of: $A_1X=\beta A_2X$ where $X=[a_1\ldots a_{N-1},b_1\ldots b_{N-1},\ldots,q_1\ldots q_{N-1}]$ is the eigenvector with A_1 and A_2 being square matrices. An important aspect of the method is its full satisfaction of the essential boundary conditions—thanks to the particular form of the base functions chosen in this work. This was done in order to avoid fast propagation of the boundary errors into the integration domain.

In the next section, we will report numerical results obtained suing the technique described above. As for the error minimization criterion, we are going to force the residue to become exactly equal to zero at N+1 collocation points. The collocation points used in this work are the Gauss-Lobatto points defined by[13],

$$y_j=\cos\left(j\frac{\pi}{N}\right);\ j=0,1,2,\ldots,N. \qquad (11)$$

In order to find the critical Taylor (or Reynolds) number, it suffices to search for the neutral instability curve. To that end, we are going to find the wavenumber for which we have: $\max[\mathrm{real}\left(\beta_{cr}(k_{cr},Re_{cr})\right)]=0.$ We have investigated the effect of the number of collocation points on the results and reached to the conclusion that by setting $N=17$ our numerical results become independent of N. So, all results to be presented shortly are for this particular number of collocation points.

RESULTS AND DISCUSSIONS

Having verified the code developed in this work using known data for Newtonian fluids, the code was then used to investigate the effect of the fluid's elasticity (as represented by the Weissenberg number) on the rise of the secondary flow. Instead of plotting the results for the critical Taylor number, we have decided to plot the results in terms of the critical inner Reynolds number defined by,

$$Re_1=\frac{\rho R_1\Omega_1 d}{\eta}. \qquad (12)$$

Figure 2 shows the effect of the Weissenberg number on the critical Reynolds number obtained at a typical mobility factor of $\alpha=0.2$ for the case in which the outer cylinder is fixed $(Re_2=0)$.

From this figure, one can conclude that fluid's elasticity has a stabilizing effect on the flow as long as it is small. But, at sufficiently high Weissenberg numbers, its effect can be destabilizing. This figure also shows that the critical Weissenberg number (i.e., the Weissenberg number beyond which elasticity starts to destabilize the flow) increases by a decrease in the gap size suggesting that for vanishingly small gap spacing, the elasticity can have a stabilizing effect on the flow over a broad range of practical Weissenberg numbers. It is also interesting to note that, at any given Weissenberg number, the critical Reynolds number is larger the smaller the gap size suggesting that working at small gaps is always preferable in Couette viscometers.

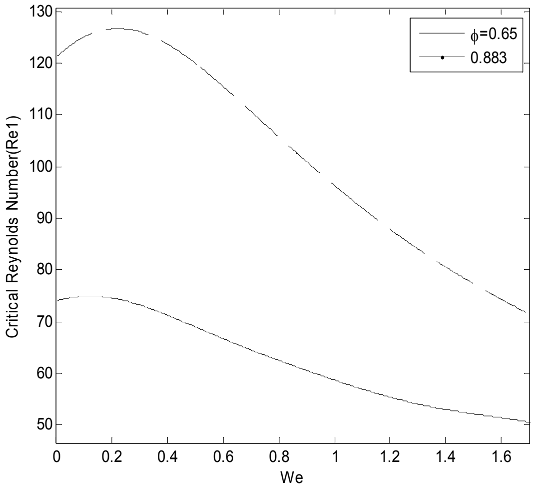

Figure 2. Effect of the Weissenberg number on the critical Reynolds number, Re_1.

Figure 3 depicts the effect of the Re_2 (i.e., the rotation of the outer cylinder) on the critical Reynolds number, Re_1, obtained at a typical Weissenberg number of We = 1.5 and a typical mobility factor of $\alpha = 0.2$ for two different gap spacing of $\varphi = 0.65$ and $\varphi = 0.883$. This figure suggests that by a decrease in the gap size, the flow becomes progressively more stable, as previously noted. Also, the rotation of the outer cylinder is predicted to have a stabilizing effect on the flow, regardless of its sense of rotation, provided that the gap spacing is sufficiently small. At large gaps, however, the smallest critical Reynolds number, Re_1, occurs when both cylinders are rotating in the same direction. For example, for $\varphi = 0.883$ Figure 3 shows that the least stable case corresponds to $Re_2 = 7$.

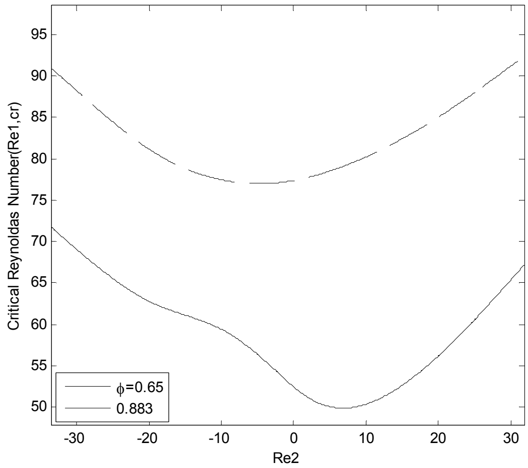

Figure 3. Effect of the rotation of the outer cylinder, Re_2, on the critical Reynolds number, Re_1, obtained at $\alpha = 0.2$, and $We = 1.5$.

CONCLUDING REMARKS

Based on the results obtained in this work, one can conclude that in Taylor-Couette flow of viscoelastic fluids, fluid's elasticity can have a stabilizing or destabilizing effect on the flow depending on the Weissenberg number being smaller or larger than a critical value. The critical Weissenberg number (i.e., the Weissenberg number beyond which, elasticity starts to destabilize the flow) is increased by a decrease in the gap size. Also, at any given Weissenberg number, the flow becomes more stable (i.e., the critical Taylor or Reynolds number is increased) by a decrease in the gap size.

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