Surface Gravity Waves on Viscoelastic Liquids

Arild Saasen

Rogaland University Centre, P.O.Box 2557, Ullandhaug, N-4004 Stavanger, Norway

ABSTRACT

Different effects of linear viscoelasticity on the propagation rate of surface gravity waves on fluids on an infinitely deep fluid layer is outlined. Emphasis is put on explaining small frequency waves where gravity dominates and high frequency waves where elasticity dominates the driving force for the waves.

INTRODUCTION

Surface gravity waves are easily observed in nature. Common to most people are the so-called water waves. These waves can be seen on sea or lake surfaces.

There are two frequently studied cases for surface gravity waves. The first case is the shallow water wave. In this case the wave length is much larger than the average water depth. For a shallow water wave the phase velocity, c_{ph}, is given by $c_{ph}=(gh)^{0.5}$, where g is the gravitational acceleration and h is the average fluid depth.

The second frequently studied case is the deep water wave problem, in which the wave length is less than twice the average fluid depth. When using this condition the fluid motion can be treated as wave motion on an infinitely thick layer of fluid.

Wave modes similar to the water wave modes may exist on the surface of an elastic body. These wave modes, normally referred to as Rayleigh-waves are frequently observed in connection with earth-quakes. These waves are non-dispersive as the phase velocity is independent of the wave length.

Earlier, water waves and Rayleigh waves were treated as separate subjects. In 1990 it was shown that deep water wave theory and Rayleigh wave theory is unified if the wave problem is treated as gravity waves on a Maxwell fluid[1]. In this note, surface gravity waves on a viscoelastic liquid with infinite depth will be discussed. Emphasis will be put on discussing wave propagation at the limit of low and high frequency of the waves, to illustrate both the water wave limit and the Rayleigh wave limit.

PROPAGATING WAVES WITH ZERO SHEAR STRESS AT THE FREE SURFACE

Waves exhibiting a spatial decay

The equations describing the fluid motion resulting from the waves are outlined in this section. The fluid velocity is described using complex potential functions. as shown in Eq. 1 and 2:

$$u = \mathrm{Re}\left(\frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial y}\right) \quad (1)$$

$$v = \mathrm{Re}\left(\frac{\partial \Phi}{\partial y} + \frac{\partial \Psi}{\partial x}\right) \quad (2)$$

The two complex potential functions are given as shown in Eq. 3 and 4:

$$\Phi = Ae^{ky}e^{i(kx-\omega t)} \quad (3)$$

$$\Psi = Be^{my}e^{i(kx-\omega t)} \quad (4)$$

In these functions (x,y) are the spatial coordinates, k is the wave number, ω is the frequency, m is a spatial damping factor determined by the equation of vorticity and i is the imaginary unit.

There are two different approaches to gravity waves. The approach outlined in this section is the development of waves exhibiting a spatial decay: The waves are generated at a point with a given frequency. The wave amplitude decreases when the wave propagates along the horizontal axis. Thus the wave amplitude is a function of the horizontal spatial coordinate only. In this approach the wave number is a function of the frequency, $k=k(\omega)$. The phase velocity of this wave is easily found since $\omega=Re(k)c_{ph}$.

The equation of vorticity give the following requirements for the spatial damping factor, m (Eq. 5):

$$m^2 = k^2 - i\frac{\rho\omega}{\mu}(1 - i\lambda\omega) \qquad (5)$$

where μ is the viscosity, ρ is the density and λ is the Maxwell fluid time constant. In the process of solving the equations of motion, the spatial damping factor, m will appear in the power of 1. To be able to use the vorticity equation this equation has to be squared. The squaring operation introduces false solutions that have to be removed.

In developing the characteristic equation, the vertical component of the equation of motion is linearized and integrated to determine the pressure as function of the potential functions. The integrated pressure equation is used in the normal stress boundary condition at the free surface, which is linearized around y=0. This normal stress equation at the free surface along with requiring the shear stress to be continuous at the free surface make a determinant that has to be zero[2].

It is convenient to make the flow calculations using dimensionless quantities. These dimensionless quantities are defined by fluid properties only, as shown in Eq. 6-9:

$$\hat{k} = \left[g\frac{\rho^2}{\mu^2} \right]^{1/3} k \qquad (6)$$

$$\hat{c} = \left[g\frac{\mu}{\rho} \right]^{1/3} c \qquad (7)$$

$$\hat{\omega} = \left[g^2\frac{\rho}{\mu} \right]^{1/3} \omega \qquad (8)$$

$$\Theta = \left[\frac{g^2\lambda^3\rho}{\mu} \right]^{1/3} \qquad (9)$$

In these Eq. 6-8 the hat (^) denotes dimensionless quantities. Since only dimensionless quantities will be discussed, the hat is dropped in the remaining part of the article. In Eq. 10 Θ is the dimensionless Maxwell fluid time constant.

The characteristic equation in dimensionless form is given by Eq. 10. In this equation the spatial decay parameter m appears in terms of m^2, only. Thus, as stated earlier, all solutions must be tested with the requirement that $Re(m)>0$.

$$0 = \sum_{n=0}^{6} B_n k^n \qquad (10)$$

where:

$B_0 = -\omega^4(1-i\theta\omega)^4$

$B_1 = 2\omega^4(1-i\theta\omega)^4$

$B_2 = -(1-i\theta\omega)^4 - 8i\omega^3(1-i\theta\omega)^3$

$B_3 = 8i\omega(1-i\theta\omega)^3$

$B_4 = 24\omega^2(1-i\theta\omega)^2$

$B_5 = -8(1-i\theta\omega)^3$

$B_6 = 16i\omega(1-i\theta\omega)$

The solution of Eq. 10 with the requirement that $Re(m)>0$ becomes the inviscid solution in dimensionless form as shown in Eq. 11 when $\omega \to 0$.

$$k = \omega^2 \qquad (11)$$

In the case where $\omega \to \infty$ it is expected that the solution is equal to the non-dispersive solution of waves demonstrating a temporal decay as dissipation is negligible.

<u>Waves exhibiting a temporal decay</u>

Within fluid mechanics research it is more common to study waves that exhibit a temporal decay. This approach gives similar results for water waves. However, when the viscosity exceeds 1 Pas and the wave length is less than 0.5 m, the results are significantly different.

Waves exhibiting a temporal decay have equal amplitude at all horizontal positions at a given time. As the waves propagate the whole sheet of waves decay equally. Thus there is a temporal decay. The temporal decay is described by slightly different potential functions than shown in Eq. 3 and 4. The potential functions used within the temporal decay theory are shown in Eq. 12 and 13.

$$\Phi = Ae^{ky}e^{ik(x-ct)} \qquad (12)$$

$$\Psi = Be^{my} e^{ik(x-ct)} \qquad (13)$$

The characteristic equation resulting from Eq. 12 and 13 is different than that from Eq. 3 and 4. In this case the phase velocity appears as the real part of c, and the damping rate appears as the imaginary part of c. Furthermore, the wave number k is now real. Thus, c is a complex function of the real parameter k: c=c(k). This is different from the spatial decay theory where the characteristic equation is given as k=k(ω), where k is complex and ω is real. As stated earlier the phase velocity can be found from both methods as shown in Eq. 14

$$c_{ph} = \mathrm{Re}(c) = \frac{\omega}{\mathrm{Re}(k)} \qquad (14)$$

The development of the characteristic equation is equal for the two different decay approaches, except for the different potential functions. The long wave solution exists in the limit $k \to 0$. For waves exhibiting a temporal decay with the requirement that Re(m)>0, the long wave solution equals the inviscid solution. This is also the case for waves exhibiting a spatial decay.

In the case of infinitely short waves, where $k \to \infty$, the solution is equal to the Rayleigh wave solution. In this case the dimensionless phase velocity is given as shown in Eq. 15.

$$c_{ph} = \beta \sqrt{\frac{1}{3\Theta}} \qquad (15)$$

where:

$$\beta = \sqrt{8 + \sqrt[3]{\sqrt{19008} - 136} - \sqrt[3]{\sqrt{19008} + 136}}$$

PROPAGATING WAVES WITH ZERO HORIZONTAL VELOCITY AT THE FREE SURFACE

The liquid content of a surface layer of a viscous fluid is often low compared to the fluid itself. This may result in the formation of a dry inelastic surface layer with no surface tension. The horizontal fluid motion at such a layer can be impossible. The boundary condition at the free surface is no longer a continuous shear stress. It is significantly simplified: The horizontal velocity component vanishes at the free surface. In this case the characteristic equation becomes as shown in Eq. 16 for the case of waves exhibiting a spatial decay, and Eq. 17 for waves exhibiting a temporal decay.

$$\left(k - \omega^2\right) m = k^2 \qquad (16)$$

$$\left(1 - kc^2\right) m = k \qquad (17)$$

These equations must be squared to be able to use the vorticity equation to remove m. In the case of temporal decay approach, the vorticity equation becomes:

$$m^2 = k^2 - ikc(1 - i\Theta kc) \qquad (18)$$

Both in case of a spatial decay approach or a temporal decay approach the solution with vanishing horizontal velocity at the free surface gives a phase velocity equal to the phase velocity on an inviscid fluid. There exists, however, no solution in the short wave limit. No solutions can be found where the fluid motion decays with depth in this case.

A Rayleigh type solution may exist for large dimensionless time constants, Θ. This solution is terminated and does not exist for frequencies greater than approximately $\Theta^{0.5}$ in the case of spatial decay approach. In the case of temporal decay approach the maximum wave number where solutions can be found is in the vicinity of k=Θ.

CONCLUSION

The phase velocity of long linear gravity waves on an infinite layer of fluid can be described by inviscid theory. If the fluid is a Maxwell fluid the infinitely short waves become Rayleigh waves if the shear stress is continuous at the free surface. If the horizontal velocity at the free surface vanishes there can not exist Rayleigh wave solutions for infinitely short waves.

REFERENCES

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2. A. Saasen, E. Kurtzhals and P.A. Tyvand, "Dispersion of Linear Gravity Waves on a Viscoelastic Fluid in an Horizontal Canal", Rheol. Acta, vol. 32, pp. 36-46, 1992.